Numerical Implementations of Holographic Duality via the Fluid/Gravity Correspondence

by

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Abstract

The fluid/gravity correspondence describes an map from relativistic fluid dynamics to general relativity in an anti de Sitter (AdS) background in one more dimension. This is a specific example of a more general principle known as holographic duality, in which a quantum field theory (QFT) is dual to a gravitational theory with the QFT defined on the boundary. Since we can regard hydrodynamics as a low-energy description of many QFTs, the fluid/gravity correspondence lets us probe holographic duality for QFTs at low energy.

In this thesis, we will discuss holographic duality, hydrodynamic theory and turbulence, numerical implementations of hydrodynamics, black branes in AdS, the fluid/gravity correspondence, and numerically testing the fluid/gravity correspondence.

Thesis Supervisor: Allan W. Adams
Title: Assistant Professor

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Chapter 1

Introduction

In this thesis, we will explore numerical implementations of the fluid/gravity correspondence, a specific example of holographic duality.

Holographic duality is a proposed equivalence between a $(d+1)$-dimensional gravitational theory and a d-dimensional quantum field theory. It was first realized in 1997 by Maldacena in [1], in what is known as the AdS/CFT correspondence. Since then, much work on AdS/CFT has been done in the theoretical physics community.

The fluid/gravity correspondence ([2] [3] [4] [5]) is a specific example of holographic duality in which the quantum field theory in d dimensions is taken in its low-energy IR description, where a fluid describes the field theory very well. The fluid/gravity correspondence equates a particular solution to relativistic hydrodynamics with the solution of a black brane in asymptotic AdS space. In particular, given any solution to hydrodynamics, we can construct a black brane in one more dimension that satisfies the Einstein Field Equations.

In 2008, Van Raamsdonk [6] explicitly wrote a $3+1$-dimensional metric given a solution to hydrodynamics in $2+1$ dimensions. In this thesis, we analytically rederive his result while correcting a few subtle and nontrivial errors in the text. We then simulate the solution by first numerically solving second order relativistic hydrodynamics, and then evaluating the metric on our hydrodynamic solution and numerically check for solving the Einstein Field Equations.

The thesis is organized as follows. Chapter 1 is the introduction. Chapter 2 provides an overview of holographic duality and the AdS/CFT correspondence, including heuristic justifications of why it is true. Chapter 3 provides an overview of relativistic hydrodynamics and the hydrodynamic expansion. Chapter 4 describes the methods we used in numerically solving relativistic hydrodynamics and provides some of our results in our simulation. Chapter 5 provides an overview of the fluid/gravity correspondence, and shows the calculation done in [6]. Chapter 6 shows our results in simulating the metric in [6]. Finally, Chapter 7 serves as a conclusion.

1.1 Conventions

We will use the mostly-pluses metric in this thesis. Lowercase Greek indices (μ, ν, $\ldots$) refer to coordinates defined in $2+1$ dimensions for the fluid on the boundary; capital Latin indices (M, N, $\ldots$) refer to coordinates defined in $3+1$ in the bulk; lowercase Latin indices (i, j, $\ldots$) refer to spatial coordinates defined for the fluid. v refers to a null-like direction.

We will use natural units in which

$$\hbar = c = k_B = 1. \tag{1.1}$$

Chapter 2

The Holographic Correspondence

2.1 Overview: AdS/CFT

The holographic correspondence is a general principle of physics that relates a d-dimensional quantum field theory to a $d+1$-dimensional gravitational theory. These two theories have equivalent partition functions, meaning that any measurable quantity (namely correlation functions) of one theory can be calculated using the other.

The holographic principle has been realized in the anti de Sitter/conformal field theory (AdS/CFT) correspondence, first discovered in 1997. Originally, Maldacena noticed that a maximally supersymmetric field theory ($\mathcal{N}=4$ supersymmetric Yang-Mills theory) was equivalent to a string theory in an AdS background in one more dimension [1].

2.1.1 Gauge Theory

Here we will give a brief review of Yang-Mills gauge theory. Much of the discussion will follow [7] [8] [9] [10]. Many of the results in this section will not be directly used, but we include it for completeness.

The main idea of a gauge theory is that we want our theory to have a *local* symmetry. Our matter fields are in some representation of the gauge group, and under a gauge transformation we insist the lagrangian remains invariant. In a Yang-

Mills gauge theory, the gauge group is defined to be the special unitary group of degree N, or $SU(N)$.

Suppose we have some field $\psi(x)$ that we decree transforms in the fundamental representation of $SU(N)$. A gauge transformation will send the field $\psi(x)$ to $U(x)\psi(x)$ where $U(x)$ is an $N \times N$ matrix in $SU(N)$ (we can think of $\psi(x)$ as an N-dimensional vector). In order to construct a dynamical field theory, however, we need to introduce derivatives. A natural first object to consider in the lagrangian is something like $\partial_\mu \psi(x)$. However, under a gauge transformation, this objects transforms as

$$\partial_\mu \psi(x) \rightarrow \partial_\mu (U(x)\psi(x)) = U(x)\partial_\mu \psi(x) + \partial_\mu U(x)\psi(x). \tag{2.1}$$

While the first term in (2.1) is what we want, the second term is awful. We define a new derivative to transform covariantly (aptly named the covariant derivative).

$$D_\mu \psi(x) \rightarrow U(x) D_\mu \psi(x). \tag{2.2}$$

In order to impose this condition, we define a new field (the gauge field) $A_\mu(x)$ in the lie algebra of $SU(N)$ such that

$$D_\mu = \partial_\mu - i A_\mu. \tag{2.3}$$

The point of the gauge field is to "cancel out" the bad term in (2.1). Thus, under a gauge transformation, we decree that A_μ transforms as

$$A_\mu \rightarrow U(x)(A_\mu + i\partial_\mu)U(x)^\dagger. \tag{2.4}$$

Note that (2.2) implies

$$D_\mu \rightarrow U(x) D_\mu U(x)^\dagger \tag{2.5}$$

which allows us to construct a gauge invariant term giving the dynamics of our gauge field. In particular, if we define $F_{\mu\nu}$ to be the commutator of two covariant derivatives, we get

$$tr(F_{\mu\nu}F^{\mu\nu}) \to tr(UF_{\mu\nu}U^{\dagger}UF^{\mu\nu}U^{\dagger}) = tr(F_{\mu\nu}F^{\mu\nu}) \tag{2.6}$$

where the trace goes over the $SU(N)$ "color" indices. This term is gauge invariant and goes as two derivatives of the gauge field. We are free to insert it into our lagrangian.

Thus we can write an $SU(N)$ Yang-Mills theory coupled to a massive fermion field $\psi(x)$ as

$$\mathcal{L} = -\frac{1}{2g^2}tr(F^{\mu\nu}F_{\mu\nu}) + \bar{\psi}(i\not{D} - m)\psi \tag{2.7}$$

The $SU(N)$ Yang-Mills theory already has a lot of symmetry – we have a local gauge symmetry at every point in spacetime. We can make this theory even more symmetric by adding sixteen super charges, making the theory what is known as the maximally supersymmetric $\mathcal{N} = 4$ supersymmetric Yang-Mills theory. This follows the physicist's generic strategy of solving problems: make the problem more and more symmetric until its tractable (aka Yang-Mills theory); if it's still too difficult, make some symmetries up (aka supersymmetry) [11] [12].

2.1.2 Gravity

We will now provide a brief review of Einstein gravity and anti de Sitter (AdS) space, following [13] [14] [15] [16].

Recall in general relativity, we describe spacetime as a manifold equipped with a metric $g_{\mu\nu}$. We can cover our manifold with a coordinate system, and our theory should remain invariant under coordinate transformation. In this sense, we can say that Einstein gravity is a gauge theory with the gauge symmetry being diffeomorphism transformation.

The equations of motion for the metric are given by Einstein field equations

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu} \tag{2.8}$$

where $R_{\mu\nu}$ is the Ricci tensor, R is the Ricci scalar, Λ is the cosmological constant, and $T_{\mu\nu}$ is the stress-energy tensor. Metric solutions in a vacuum satisfy

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = 0 \tag{2.9}$$

In this paper we will be particularly interested in a solution of Einstein gravity known an anti de Sitter space.

Consider the metric

$$ds^2 = L_{AdS}^2 \frac{-dt^2 + d\vec{x}^2 + dr^2}{r^2} \tag{2.10}$$

This is the solution for d-dimensional anti de Sitter spacetime in a coordinate system known as Poincaré coordinates. L_{AdS} is the radius of curvature of AdS space, a dimensionful quantity relating to the curvature of the AdS space. The Ricci scalar of the space is given by

$$R = -\frac{d(d-1)}{L_{AdS}^2} \tag{2.11}$$

and the metric is only consistent with a cosmological constant given by

$$\Lambda = -\frac{(d-1)(d-2)}{2L_{AdS}^2}. \tag{2.12}$$

Note that as L_{AdS} goes to ∞, the curvature vanishes and the space approaches flat. This makes sense – if the relevant length scale of some system is much smaller than the relevant length of scale of spacetime, the curvature will go unnoticed and the spacetime will appear flat.

AdS space is a homogenous, isotropic space of constant negative curvature. It is equivalent to a hyperbolic space, but with a $(-,+,+,\ldots,+)$ metric signature, in the same way that dS space is equivalent to a sphere but with a different signature.

Black Branes in asymptotic AdS

Motivated by a Schwarzschild black hole, we can modify our metric to be of the form

$$ds^2 = \frac{L_{AdS}^2}{r^2}(-f(r)dt^2 + d\vec{x}^2 + \frac{1}{f(r)}dr^2) \tag{2.13}$$

This metric solves the Einstein field equations in vacuum if

$$f(r) = 1 - \frac{r^{d-1}}{r_H^{d-1}} \tag{2.14}$$

where r_H is any constant. It describes AdS with a black brane with horizon at a surface located at $r = r_H$.

Under a coordinate transformation, we can rewrite this metric as

$$ds^2 = 2dvdr - r^2 f(r) dv^2 + r^2 d\vec{x}^2 \tag{2.15}$$

where $f(r) = 1 - \frac{r_H^{d-1}}{r^{d-1}}$. These coordinates are known as infalling coordinates.

Reissner-Nordstrom Black Branes in asymptotic AdS

This section will follow mostly [17].

Let us consider adding a electromagnetic vector potential field to our AdS space, with a component $A_t(r)$ but otherwise zero. This will become a charged black hole in asymptotic AdS space.

This is solution is valid when $f(r)$ has a form that goes as

$$f(r) = 1 - (d-2)Mr^{d-1} + Q^2 r^{2d-4} \tag{2.16}$$

where M and Q are suggestively named coefficients. It can be shown that M and Q correspond to the mass density and charge density respectively of the black brane.

Note that $f(r)$ vanishes generically at two points – these correspond to two horizons in a generic Reissner-Nordstorm black brane.

When reasons discussed in Sec 2.2.2 (though we will not explicitly do the calculation), we will associate the derivative of f at the outer horizon (closer to the boundary), $f'(r_+)$ with the temperature of the black brane. When this quantity vanishes is also when the black brane has only one horizon, which can be seen in Fig. 2-1. This is known as an extremal Reissner-Nordstrom black brane, one in which the temperature vanishes.

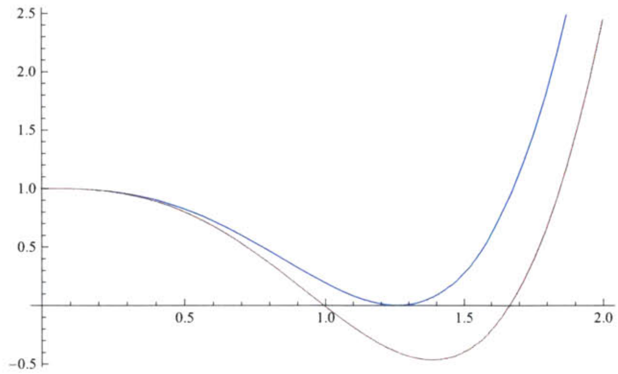

Figure 2-1: Plot of two different functions $f(r)$ as a function of r. The blue corresponds to an extremal Reissner-Nordstrom black brane, with zero temperature and only one horizon; the red corresponds to a non-extremal Reissner-Nordstrom black brane, with nonzero temperature and two horizons.

In such an extremal RN black hole, we can show that

$$M = CQ^{\frac{d-1}{d-2}} \tag{2.17}$$

where C is a dimension-dependent constant, specifically

$$C = \frac{1 + \frac{d-1}{d-3}}{(d-2)\left(\frac{d-1}{d-3}\right)^{\frac{d-1}{2d-4}}}. \tag{2.18}$$

The interpretation of this result is quite beautiful. Any object with nonzero temperature will radiate energy via blackbody radiation, including a black hole. For a black hole, this is known as Hawking radiation. A RN black brane will radiate away matter until the temperature becomes zero, at which point its mass and charge are equal (in suitable units). But if the temperature is not zero, then the black brane has, roughly

$$M > Q. \tag{2.19}$$

Because we have quantized charge, we can never have matter with charge greater than its mass. Thus our black brane can radiate matter (which will have $M > Q$) away until it approaches extremality (when its mass and charge are equal).

The holographic dual of a charged system will be a RN black hole; the holographic dual of a system with nonzero temperature will be a black brane with nonzero temperature. Although we will not use RN black holes in this thesis, it is a beautiful bit of physics that deserves to be described briefly.

2.1.3 Duality

In this section we will return to the story of holographic duality, and note why it is not only very interesting but also a useful idea to study, following [17] [18]. When AdS/CFT was first discovered by Maldacena, it was a duality between $\mathcal{N} = 4$ supersymmetric Yang-Mills theory and a type of string theory in $AdS_5 \times S^5$.

The two parameters describing the Yang-Mills theory are (1) the coupling constant of the theory g, and (2) the value of N for an $SU(N)$ Yang-Mills gauge theory. As g grows large[1] the theory becomes nonperturbative and calculational tools using perturbative quantum field theory (e.g. Feynman diagrams) become useless. As N grows large, the number of degrees of freedom at every point grows roughly quadratically (this does not necessarily make the theory intractable, see [19] for example). In fact, it is only because this theory is so symmetric that such detailed calculations in it are possible; generically physicists are quite bad at calculating quantities in strongly-coupled theories.

On the other hand, the two dimensionless parameters describing the string theory are (1) the ratio of the length of the string, l_s to L_{AdS}, and (2) the ratio of the Planck length l_p to L_{AdS}. The parameter l_s determines how important the role strings play in the description of gravity. At length scales much larger than it, we can integrate out the stringy effects. Likewise, the parameter l_p determines the effect of quantum corrections to our gravity theory. When either of these quantities are large, the theory becomes much more difficult to understand.

Maldacena found in the first example of AdS/CFT that a supersymmetric Yang-

[1] We of course need to be careful what we mean by this. g is a coupling constant and therefore has some flow along the renormalization group, so it is not entirely clear what we mean by "large" g, but we are being schematic here.

Mills theory described by some g and N was equivalent to a string theory with

$$l_s \sim \frac{1}{g} \tag{2.20}$$

and

$$l_p \sim \frac{1}{N}. \tag{2.21}$$

In more recent discoveries, the same general trend has been noticed. As the quantum effects of the gravity theory ($\sim l_p$) grows, the number of degrees of freedom of the gauge theory ($\sim N$) shrinks; and as the stringy effects of the gravity theory ($\sim l_s$) grows, the coupling constant of the gauge theory ($\sim g$) shrinks. Thus roughly, as one theory becomes more difficult, the other becomes easier.

This makes AdS/CFT a powerful calculational tool in addition to saying something very deep about gauge theories and gravitational theories. Instead of trying to calculate correlation functions in a strongly-coupled quantum field theory, we can look at the dual theory instead. If the field theory is very strongly coupled and has many degrees of freedom, the dual theory will reduce to a gravitational theory where the string length and the Planck length can be integrated out. This is none other than classical Einstein gravity.

AdS/CFT has made much progress in studying the physics of QCD and quark-gluon plasmas [18] [20], entanglement entropy [21], high temperature superconductivity [22], phase transitions, and fluid mechanics [3]. This is a truly amazing fact – physicists are working on strange metals and high temperature superconductivity by thinking about black branes in asymptotically AdS space. In the second half of this paper, we will focus on applications of holography to fluid mechanics and hydrodynamics.

2.2 Motivation

The AdS/CFT correspondence makes a bold and shocking claim: the information in a gravitational theory in $d+1$ dimensions can be reorganized and repackaged into a an equivalent field theory in d dimensions. In this section, I will give a few heuristic arguments as to why this equivalence had to be true.

2.2.1 Black Holes and Observers

This discussion will mostly follow [17].

Imagine a black hole in some region of space. An intrepid farmer (who lives in $3+1$ dimensions) throws a constant stream of goats uniformly around the black hole. The physics of the system has to be the same regardless of the observer – be it the farmer or his goats.

The farmer will see a steady stream of goats heading towards the event horizon, upon which they will slow down and spread across the black hole. Their signals will become redshifted as they form a surface on the event horizon, eventually no longer sending any new information to the farmer. According to him, the goats have stopped moving and have instead spread across the event horizon. They form a $2+1$ dimensional fluid without any gravitational effects. Perturbing the system (e.g. throwing another goat) will cause ripples to form across the fluid's surface that will eventually dissipate, just as in fluid dynamics. In fact, the farmer can model the entire system using dissipative fluid mechanics (he's a very educated farmer).

But what do the goats see? They charge straight towards the black hole and pass through the event horizon unscathed (the curvature of a black hole is all located at its singularity, so classical general relativity should serve as a fine approximation for the physics at the event horizon). Eventually, they reach the gravitational singularity, where they are presumably shuffled off this mortal coil.

The goats can describe the physics of their situation as a gravitational theory in $3+1$ dimensions. However, the farmer as we discussed earlier sees everything as a nongravitational theory in $2+1$ dimensions. Somehow the physics of these two startlingly different theories must be equivalent (perhaps in a deeply nonlocal and complex way).

2.2.2 Black Hole Thermodynamics

Another way we can heuristically justify AdS/CFT is through a consideration of black hole entropy and thermodynamics. We will mostly follow [23] [24] in this discussion.

The physics of black holes has many provocative parallels to the laws of thermodynamics which we will discuss here.

Black Hole Entropy

A theorem of classical general relativity, the no-hair theorem, states that a black hole in asymptotically flat spacetime is described by only three parameters: mass, charge, and angular momentum. All of the information of what matter formed the black hole must therefore be stored near the singularity of the black hole, where quantum corrections to classical Einstein gravity can violate the no-hair theorem. But this is inside the event horizon and therefore inaccessible and causally disconnected to observers.

Entropy measures the amount of ignorance we as observers have of a system. A black hole is the perfect "black box" for entropy – by the no-hair theorem, an observer has no idea what original matter went into forming the black hole. A black hole should contain enormous amounts of entropy.

In 1972, Bekenstein showed the entropy of a black hole (measuring the number of degrees of freedom) should scale as its area, rather than its volume.

$$S = \frac{A}{4l_p^2} \tag{2.22}$$

Laws of Black Hole Thermodynamics

By analogy to thermodynamics, a change in the mass of the black hole is given by

$$dM = \frac{\kappa}{8\pi} dA \tag{2.23}$$

where κ is the surface gravity of the black hole at its event horizon. We thus associated the temperature of a black hole to $\frac{\kappa}{2\pi}$. We now present the laws of black hole thermodynamics; their parallels to the laws of thermodynamics should be clear.

0. A black hole in equilibrium has constant surface gravity on its horizon.

1. The change in mass (energy) of a black hole is given by

$$dM = \frac{\kappa}{8\pi} dA + \dots \tag{2.24}$$

   where ... include other parameters of the black hole, such as angular momentum, charge, etc.

2. The area of a black hole never decreases.

$$\frac{dA}{dt} \geq 0 \tag{2.25}$$

3. In finite time, black hole will never have zero surface gravity.

The laws of thermodynamics are recovered when temperature replaces surface gravity and entropy replaces area.

Maximal Entropy: The Bekenstein Bound

An interesting consequence arises from black hole thermodynamics. Given any region of space, we can calculate the entropy it would have if it were a black hole.

Now suppose we take matter and compress it into this region. As we keep adding more and more matter, the entropy of the system will increase. But eventually we will have enough matter to form a black hole, at which point we know the entropy. In order to satisfy the second law of thermodynamics, the entropy of the black hole must be larger than the entropy of the matter before collapsing.

Thus we conclude that the maximum entropy a region of space can have is that of a black hole in that space. This is known as the Bekenstein Bound, limiting the amount of information that can be stored in a particular space. If this bound were violated, a perfectly efficient Carnot engine could be built by dumping the excess heat into a black hole.

But this is a strange result. We know that black hole entropy scales as its area, whereas normally we think of entropy as an extrinsic property, scaling as volume. This is a hint towards holography, in which we associate the black hole with an extra dimension compared. If we do this, then the entropy scaling for both black holes and ordinary matter are equal.

2.2.3 Geometrizing RG flow

Here we will provide one final interpretation of AdS/CFT and holographic duality, through a visualization of renormalization group (RG) flow. We again follow the argument in [17] [18].

Recall the idea of Wilsonian effective field theories, in which physics is organized by length scales. UV physics at length scales very small are integrated over, and an effective theory is constructed. We can then imagine a "flow" of some coupling constant along the physics of different length scales.

A canonical example is through physics defined on a lattice. Imagine a theory defined on some lattice with spacing a. A generic Hamiltonian of the theory will consist of a bunch operators $\mathcal{O}$ and their sources (coupling constants) J at points on the lattice, x.

$$H = \sum_{x,i} J_i(x)\mathcal{O}_i(x) \tag{2.26}$$

Wilson then considered looking at this theory at a lower energy, by effectively "coarse-graining" the lattice. We can define a new lattice with distance $2a$, for instance, and average over the physics of the four sites. The couplings $J_i(x)$ can be tuned so that the physics of the low-energy modes are the same.

We have now defined a new set of couplings, $J_i(x, r)$ where r is the length scale on the lattice (for instance a or $2a$). The key point in the renormalization group is that the change in J_i along r is *local*, meaning it satisfies some local equation

$$r\frac{\partial}{\partial r}J_i(x,r) = \beta(J_i, r). \tag{2.27}$$

This is known as the beta function of some coupling J, which we can often perturbatively solve for by considering, e.g., divergent loop diagrams in perturbative quantum field theory.

It is tempting to associate the source $J_i(x,r)$ with a field in one more dimension, the extra dimension being the RG scale r. We can attempt to define a field theory in one more dimension ("in the bulk") with bulk fields $\Phi_i(x,r)$ that reduce to the sources in the UV limit. Namely,

$$\Phi_i(x, r=a) = J_i(x). \tag{2.28}$$

Thus we have a one-to-one correspondence between bulk fields Φ_i, and couplings J_i on the boundary.

Consider, on the boundary theory, a stress-energy operator $T^{\mu\nu}(x)$. This has a coupling with a rank-two tensor structure. Thus, in the bulk, we must have a spin-two field associated with the coupling, which we will suggestively call $g_{\mu\nu}(x,r)$. By various theorems by Weinberg and Witten, it can be shown that this spin-two field must be considered as the metric in general relativity. We can therefore compute any correlation function as a field theory on the boundary, or as a gravity problem in the bulk.

In fact, the geometry of AdS space lends itself to a very nice interpretation of the RG flow. Recall the Poincaré metric for AdS space, given in (2.10). At a constant r, this is none other than flat Minkowski space. But if we vary r, it appears as though the space is stretched by some overall scaling factor. Thus the direction r in AdS space is the same r we used in our discussion of RG flow. AdS space provides a very convenient visualization of the flow of length scales.

2.3 Why AdS?

Although holography was first realized with an AdS space, this does not mean de Sitter or flat spacetimes lack holographic duals. In fact, a lot of work right now is being done on finding a holographic dual to de Sitter space, or dS/CFT, see [25].

The reason physicists discovered AdS/CFT first is because in many ways, anti de Sitter space is a playground for physics. If a system is very complicated to study, a common trick to put the theory in a box, with sufficient boundary conditions at the edges. But this cannot work with gravity – we cannot impose boundary conditions in general relativity because our dynamical field is the metric itself.

The miracle of AdS space is we *can* do this with AdS. Although the space is infinite, we have a boundary at $z = 0$, and all null geodesics will eventually return to their spatial location. If you throw a rock in AdS space, it will eventually come back to you. AdS space is like a box. The reason holographic duality was first discovered with AdS/CFT is not because these dualities are somehow more deep than duals of dS or flat spaces, but because AdS spaces are much easier to deal with.

Chapter 3

Fluid Mechanics and the Hydrodynamic Expansion

In this section, we will discuss relativistic hydrodynamics at zeroth, first, and second order. For other introductions to hydrodynamics, see [26] [27].

3.1 Hydrodynamic Expansion

Hydrodynamics is a field theory description of a fluid in flat space based on only one "fundamental field", the fluid velocity $u^\mu(x)$ as a function of the spacetime position of the fluid. Recall that u^μ is simply the relativistic four-velocity, which in component form can be written as

$$u^\mu = \frac{1}{\sqrt{1 - v_x^2 - v_y^2}} \begin{pmatrix} 1 \\ v_x \\ v_y \end{pmatrix}. \tag{3.1}$$

From the fluid velocity, we will construct a conserved Noether current $T^{\mu\nu}$ corresponding to spacetime translation invariance. This current is known as the stress-energy tensor. The basic assumption used in hydrodynamics is known as the *hydrodynamic expansion* in which we assume that variation of the velocity field is small. In other words, we assume roughly

$$u >> \partial u >> \partial^2 u >> \ldots \tag{3.2}$$

where $\partial^n u$ is a Lorentz covariant quantity that scales as n derivatives of the field. From this hypothesis, we can perform a series expansion in derivatives of the field and solve the theory perturbatively. More formally, we introduce a scale parameter ϵ, and reparametrize our spacetime coordinates by ϵ. Explicitly, we write

$$u(t, \vec{x}) \to u(\epsilon t, \epsilon \vec{x}) \tag{3.3}$$

perform a series expansion in ϵ in our stress-energy tensor. In n^{th} order hydrodynamics, we keep all terms to order ϵ^n. After dropping appropriate terms in ϵ, we finally set it to unity.

Note that because we are working in the relativistic limit and $v \sim 1$, we have $t \sim \vec{x}$, so both parameters scale as ϵ. We will discuss briefly what happens in the nonrelativistic limit in section 3.2.

The equations of motion of hydrodynamics are simply given by the conservation of our Noether current to some order in our formal parameter ϵ (or equivalently by the chain rule, to some order in derivative)

$$\partial_\mu T^{\mu\nu} = 0. \tag{3.4}$$

3.1.1 Zeroth Order Hydrodynamics

In zeroth order hydrodynamics, our stress-energy tensor contains no terms with any derivatives on u^μ. Thus the only two terms with the right index structure we can put in our tensor are $u^\mu u^\nu$ and our metric tensor, $\eta^{\mu\nu}$.

Let's define two parameters ρ and p so that our stress-energy tensor is of the form

$$T_0^{\mu\nu} = (\rho + p) u^\mu u^\nu + p\eta^{\mu\nu}. \tag{3.5}$$

Note that in the local rest frame of the fluid, we have that[1]

$$T_0^{\mu\nu} = \begin{pmatrix} \rho & 0 & 0 \\ 0 & p & 0 \\ 0 & 0 & p \end{pmatrix}. \tag{3.6}$$

Recall the interpretation of the stress-energy tensor as the flux of four-momentum in a surface of constant four-direction.[2] We can therefore, in the rest frame of the fluid, associate ρ with the energy density and p with the pressure.

We would really like now an equation of state relating the thermodynamic quantities of ρ and p. Recall for an ultra-relativistic system, the energy goes roughly as the momentum

$$E \sim |p| \tag{3.7}$$

This then implies our equation of state

$$\rho = 2p \tag{3.8}$$

and therefore the stress-energy tensor goes as

$$T_0^{\mu\nu} \sim \eta^{\mu\nu} + 3u^\mu u^\nu \tag{3.9}$$

From dimensional analysis, T_0 is proportional to energy density, which in $2+1$ dimensions has mass dimension 3. Thus, our zeroth order stress-energy tensor is[3]

$$T_0^{\mu\nu} = T^3(\eta^{\mu\nu} + 3u^\mu u^\nu) \tag{3.10}$$

where T is the temperature of the fluid.

The equations of motion of the fluid are most elegantly and naturally written in

[1] As a reminder, we are working in $2+1$ dimensions. The generalization to $3+1$ or $d+1$ dimensions is straightforward.

[2] Here four-momentum refers to a Lorentz covariant vector. For us, the four-momentum is of course a three-dimensional vector.

[3] Up to numerical factors which are conventions.

the Lorentz covariant form of

$$\partial_\mu T_0^{\mu\nu} = 0. \tag{3.11}$$

Of course, if we wanted to, we could write these equations out in terms of their components ruining the Lorentz covariance. We will discuss more about numerical implementations of solving hydrodynamics in Section 4. Equations (3.11) are known as the relativistic Euler equations, and model liquids without any viscosity or dissipation.

3.1.2 First Order Hydrodynamics

In order to model dissipative fluids, we need to modify our stress energy tensor by including more terms in our perturbative expansion. However, we need to be careful: we would like to preserve all of the symmetries currently in place, as well as our definitions of global quantities, such as the energy. Let's define a tensor $\Pi^{\mu\nu}$ to be all higher-order terms in our stress-energy tensor, or

$$T^{\mu\nu} = T_0^{\mu\nu} + \Pi^{\mu\nu} \tag{3.12}$$

and see what constraints on $\Pi^{\mu\nu}$ we must impose.

First of all, $\Pi^{\mu\nu}$ must transform as a rank two tensor under Lorentz transformation. Thus it must be built covariantly out of the fluid velocity u^μ, and the metric $\eta^{\mu\nu}$.

Next, we define the stress-energy tensor to be symmetric. Thus

$$\Pi^{\mu\nu} = \Pi^{\nu\mu}. \tag{3.13}$$

We also would like to have a Lorentz-invariant definition of the rest energy. In zeroth order hydrodynamics, we know this is

$$u_\mu u_\nu T_0^{\mu\nu} = \rho \tag{3.14}$$

by looking at the local rest frame. In order to preserve this definition, $\Pi^{\mu\nu}$ must satisfy

$$u_\mu \Pi^{\mu\nu} = 0. \tag{3.15}$$

Finally, we require conformal symmetry. This means our theory is scale independent, which implies that

$$\delta g_{\mu\nu} = \Lambda g_{\mu\nu} \tag{3.16}$$

is a symmetry of our theory and leave the action invariant. Since the stress-energy tensor is the source of the metric, this means

$$\Lambda g_{\mu\nu} T^{\mu\nu} = 0 \tag{3.17}$$

or equivalently

$$T^\mu_\mu = 0. \tag{3.18}$$

Since the zeroth order stress-energy tensor is already traceless, we get our final constraint of $\Pi^{\mu\nu}$:

$$\Pi^\mu_\mu = 0. \tag{3.19}$$

Note that this is a requirement we imposed for conformal fluids, which do not describe all fluid systems. We restrict ourselves to conformal fluids in order to make the calculations much easier, but in principle we could allow for nonconformal fluids. The analysis would be different and more difficult, so we will not attempt to do so here.

To summarize, we need $\Pi^{\mu\nu}$ to be traceless, symmetric, and orthogonal to u^μ. First-order hydrodynamics has the equations of motion for a stress energy tensor with all consistent terms that go as one derivative and satisfy the constraints in equations (3.13), (3.15), and (3.19).

We define *first-order hydrodynamics*, for instance, to be the hydrodynamics of a theory including all terms in the stress-energy tensor that have zero or one derivative of the fluid, u^μ. The coefficients in front of each term in $\Pi^{\mu\nu}$ are phenomenologically determined constants called *transport coefficients*.

If we are given an arbitrary tensor $A^{\mu\nu}$, we can construct a tensor that satisfies equations (3.13), (3.15), and (3.19) by applying various procedures.

In order to construct a symmetric tensor, we extract out the symmetric component of $A^{\mu\nu}$:

$$A^{\mu\nu}_{\text{sym}} = A^{\mu\nu} + A^{\nu\mu} \tag{3.20}$$

To force $A^{\mu\nu}$ to be orthogonal to u^{μ}, we first define a projector

$$P^{\mu\nu} = \eta^{\mu\nu} + u^{\mu}u^{\nu} \tag{3.21}$$

which satisfies

$$u_{\mu}P^{\mu\nu} = 0. \tag{3.22}$$

Thus it projects vectors to the orthogonal subspace of u^{μ}. If we act on $A^{\mu\nu}$ with the projector operator, we can get an orthogonal piece to u^{μ}.

$$A^{\perp\mu\nu} = P^{\mu}_{\lambda}A^{\lambda\nu}. \tag{3.23}$$

Finally, to make $A^{\mu\nu}$ traceless, we simply subtract off the trace.

$$A^{\mu\nu}_{\text{traceless}} = A^{\mu\nu} - \frac{1}{d}A^{\lambda}_{\lambda}\eta^{\mu\nu}. \tag{3.24}$$

If we apply (3.20), (3.23), and (3.24) all at once, we get a projector onto traceless, symmetric tensors orthogonal to u^{μ}. This is of the form

$$\Pi^{\mu\nu}_{\alpha\beta} = \frac{1}{2}P^{\mu}_{\alpha}P^{\nu}_{\beta} + \frac{1}{2}P^{\mu}_{\beta}P^{\nu}_{\alpha} - \frac{1}{2}P^{\mu\nu}P_{\alpha\beta}. \tag{3.25}$$

The only $A^{\mu\nu}$ we can build in first-order hydrodynamics is $\partial^{\mu}u^{\nu}$, so we therefore only have one valid piece in $\Pi^{\mu\nu}$. By definition, it is

$$\sigma^{\mu\nu} = \Pi^{\mu\nu}_{\alpha\beta}(\partial^{\alpha}u^{\beta}) \tag{3.26}$$

Thus a first order stress-energy tensor is of the form

$$T^{\mu\nu} = \xi_1 T^3(\eta^{\mu\nu} + 3u^\mu u^\nu) + \xi_2 T^2 \sigma^{\mu\nu} \tag{3.27}$$

where ξ_1 and ξ_2 are transport coefficients. Note that the coefficient in front of $\sigma^{\mu\nu}$ scales as T^2 from dimensional analysis.

First order hydrodynamics has dissipative terms, namely $\sigma^{\mu\nu}$. However, it is numerically unstable as we will discuss in Section 4.2.

3.1.3 Second Order Hydrodynamics

If we perform the same procedure, but this time allow for two derivatives on the fluid velocity field, we arrive at the most general metric in (3.28).

$$T^{\mu\nu} = T^3\xi_1(\eta^{\mu\nu} + 3u^\mu u^\nu) + T^2\xi_2\sigma^{\mu\nu} + T\xi_3\Sigma_1^{\mu\nu} + T\xi_4\Sigma_2^{\mu\nu} \tag{3.28}$$

with T defined as the temperature and

$$\omega^{\mu\nu} = \frac{1}{2}P^\mu_\alpha P^\nu_\beta(\partial^\alpha u^\beta - \partial^\beta u^\alpha) \tag{3.29}$$

$$\Sigma_1^{\mu\nu} = \Pi^{\mu\nu}_{\alpha\beta}(\mathcal{D}\sigma^{\alpha\beta} + \frac{1}{2}\sigma^{\alpha\beta}\partial_\lambda u^\lambda) \tag{3.30}$$

and

$$\Sigma_2^{\mu\nu} = \Pi^{\mu\nu}_{\alpha\beta}(\sigma^\alpha_\lambda\omega^{\beta\lambda}). \tag{3.31}$$

$\mathcal{D}$ is defined to be $u^\alpha\partial_\alpha$, a convective derivative.

3.2 Non-Relativistic Limit

The equations of motion given in (3.4) are relativistic equations. If our fluid has nonrelativistic speeds, meaning

$$v << 1 \tag{3.32}$$

then our equations of motion can simplify. Formally, the nonrelativistic limit means our scaling will be different. In the nonrelativistic limit,

$$t \sim \frac{1}{E} \sim \frac{1}{v^2} \sim \frac{1}{p^2} \sim x^2. \tag{3.33}$$

Thus if we send

$$x \to \epsilon x \tag{3.34}$$

then we have

$$x^2 \to \epsilon^2 x^2 \tag{3.35}$$

or

$$t \to \epsilon^2 t. \tag{3.36}$$

Thus our scaling is

$$u(t, x, y) \to u(\epsilon^2 t, \epsilon x, \epsilon y) \tag{3.37}$$

in the nonrelativistic limit.

If we take the nonrelativistic limit of first order hydrodynamics, we obtain the famous Navier-Stokes equations:

$$\rho\left(\frac{\partial v}{\partial t} + v \cdot \nabla v\right) = -\nabla p. \tag{3.38}$$

Note that because we have removed the explicit Lorentz invariance, these equations are somewhat more unwieldy than (3.4).

3.3 Turbulence

The fundamental assumption of hydrodynamics is the hydrodynamic scaling argument in (3.2). If this postulate breaks down, then we start doing a perturbative expansion in a large parameter and our model is no longer valid. This happens in a phenomenon known as *turbulence* and is still a relatively open problem in classical physics.

3.3.1 Kolmogorov Scaling

In 1941, Kolmogorov made one of the few arguments about nonrelativistic turbulence that have been discovered, known as Kolmogorov scaling. In a turbulent system, the only relevant parameters are the energy dissipation of the system, and the wavenumber of a mode we are interested in. Thus, by dimensional analysis we can see what the energy density in a particular mode should scale as. Note that we are considering all quantities as densities, so no mass parameters show up.

The units of energy density per mass per wavelength are

$$E(k) \sim \frac{\text{length}^3}{\text{time}^2} \tag{3.39}$$

and the units of energy dissipation are

$$\epsilon \sim \frac{\text{length}^2}{\text{time}^3}. \tag{3.40}$$

Finally the wavenumber units are of course

$$k \sim \frac{1}{\text{length}} \tag{3.41}$$

Thus the scaling of energy must go as

$$E(k) \sim \epsilon^{\frac{2}{3}} k^{-\frac{5}{3}} \tag{3.42}$$

known as Kolmogorov scaling, or Kolmogorov's "$\frac{5}{3}$ law" for nonrelativistic scaling.

Remarkably this scaling has been seen experimentally to quite high precision and generality, yet a derivation of the result without using dimensional analysis is unknown. Moreover, Kolmogorov's 70-year-old result is one of the only definite results physicists have for turbulent systems.

This result has been seen for ideal relativistic hydrodynamics in [28] and [29].

Chapter 4

Numerical Methods in Solving Hydrodynamics

4.1 Zeroth Order Hydrodynamics

For zeroth order hydrodynamics, writing a numerical solution is simple enough. The equations of motion are given, in their relativistic form, in (3.11). But as we alluded to earlier, in order to code the solver up, we need to break the Lorentz invariance and write the equation component by component. The relevant variables we have are T, u_x, and u_y, and (3.11) is a first-order differential equation in terms of these variables. We can rewrite the three equations in (3.11) and rearrange them using Mathematica to get

$$
\begin{aligned}
\dot{T} &= \frac{-(\partial_x T + \partial_y T)(1+u_x^2+u_y^2) - T(\partial_x u_x(1+u_y^2) - u_x u_y(\partial_y u_x + \partial_x u_y) + \partial_y u_y(1+u_x^2))}{\sqrt{1+u_x^2+u_y^2}(2+u_x^2+u_y^2)} \\
\dot{u_x} &= \frac{1}{T\sqrt{1+u_x^2+u_y^2}(2+u_x^2+u_y^2)} \\
&\quad (-(\partial_x T + T\partial_y u_x u_y)(2+u_y^2) - u_x^2(2\partial_x T + T u_y(2\partial_y u_x + \partial_x u_y)) \\
&\quad -u_x(T\partial_y u_y + T(\partial_x u_x - \partial_y u_y)) + T u_x^3(-\partial_x u_x + \partial_y u_y)) \\
\dot{u_y} &= \frac{1}{T\sqrt{1+u_x^2+u_y^2}(2+u_x^2+u_y^2)} \\
&\quad (-(\partial_y T + T\partial_x u_y u_x)(2+u_x^2) - u_y^2(2\partial_y T + T u_x(2\partial_x u_y + \partial_y u_x)) \\
&\quad -u_y(T\partial_x u_x + T(\partial_y u_y - \partial_x u_x)) + T u_y^3(-\partial_y u_y + \partial_x u_x)).
\end{aligned} \qquad (4.1)
$$

Every term on the lefthand side of (4.1) is a time-derivative; every term on the righthand side contains only the variables and spatial derivatives. Thus, a solution of zeroth order hydrodynamics can be found by choosing arbitrary initial conditions on a grid, and time evolving by calculating the numerical values for $\dot{T}$, $\dot{u_x}$, and $\dot{u_y}$. When given the numerical value of $\dot{T}$, $\dot{u_x}$, and $\dot{u_y}$, we can simply add it to our current values to obtain the next time step.

In order to efficiently calculate the spatial derivatives on the grid, we used spectral interpolation with a Fourier basis in MATLAB. See Appendix B for further discussion on taking derivatives on a finite grid. All spatial derivatives were taken using spectral decomposition, while time derivatives were done with finite differencing.[1] We used periodic boundary conditions in a finite box.

4.2 First Order Hydrodynamics

In first order hydrodynamics, the stress-energy tensor consists of terms up to one derivative in velocity field. Thus, the conservation equation (3.4) will have terms containing up to two derivatives in u^μ. Thus, there will be terms of up to two time-derivatives in the equations of motion. In temperature, though, there will still be at

[1]Thanks to Allan Adams and Paul Chesler for help in setting up spectral interpolation.

most one time derivative. We can write the equations schematically as:

$$\begin{aligned}
\dot{T} &= a(T, u_x, u_y, v_x, v_y) \\
\dot{v_x} &= b(T, u_x, u_y, v_x, v_y) \\
\dot{v_y} &= c(T, u_x, u_y, v_x, v_y) \\
\dot{u_x} &= v_x \\
\dot{u_y} &= v_y
\end{aligned} \tag{4.2}$$

where a, b, and c are functions determined from (3.4). Note that we defined $\dot{u_x}$ and $\dot{u_y}$ to be v_x and v_y, so the functions a, b, and c should have no time derivatives in them. Any instances of one time derivative on u^μ can be replaced with an appropriate v_x or v_y; any instance of two can be moved to the left hand side of the equation.

Given any set of initial conditions, including an appropriate v_x and v_y, we can simulate first order hydrodynamics. However, relativistic first order hydrodynamics is unstable. Numerical deviations from the "true" solution, rather than vanishing at later time steps, blow up and quickly render the system unphysical. We will therefore instead of studying numerical implementations of first order hydrodynamics, look to second order (which is numerically stable).

4.3 Second Order Hydrodynamics

The generalization of (4.2) for second order hydrodynamics is

$$\begin{aligned}
\dot{T} &= a(T, u_x, u_y, v_x, v_y, w_x, w_y) \\
\dot{w_x} &= b(T, u_x, u_y, v_x, v_y, w_x, w_y) \\
\dot{w_y} &= c(T, u_x, u_y, v_x, v_y, w_x, w_y) \\
\dot{v_x} &= w_x \\
\dot{v_y} &= w_y \\
\dot{u_x} &= v_x \\
\dot{u_y} &= v_y
\end{aligned} \tag{4.3}$$

In practice, however, it becomes very quickly intractable to solve for a, b, and c since the equations of second order hydrodynamics become extremely complicated. However, we can simplify the equations through several methods.

First note that we are working perturbatively to second-order in derivative. We will use this fact to eliminate some of our variables by rewriting our equations.

In second order hydrodynamics, $T^{\mu\nu}$ is defined only to second order in derivative. Suppose we explicitly solved 3.4 for $\ddot{u_x}$ and $\ddot{u_y}$ in terms of variables with one or fewer time derivative. If we then plugged the result back into our expression for $T^{\mu\nu}$, it would have terms that go as three derivatives of u.[2] But we can discard those terms because we only define $T^{\mu\nu}$ to second order.

This procedure would remove all instances of $\ddot{u_x}$ and $\ddot{u_y}$, and reduce our seven variables to five. Note that the same trick could have been used to solve first order hydrodynamics, reducing five variables to three.

Our next move will be to change our variables to more convenient and natural ones. $\dot{u_x}$ and $\dot{u_y}$ (or as we called them in (4.3), v_x and v_y) are manifestly nonrelativistic and break the symmetric structure of our original equations of motion. Although any numerical implementation and choice of variables must break the symmetry, we can

[2]Some of these derivatives would be spatial.

choose more natural variables. We also want, for numerical convenience, variables that lead to non-degenerate solutions, meaning that our equations of motion do not contain terms that blow up at zero velocity.

Recall that we wrote our stress energy tensor as

$$T^{\mu\nu} = T_0^{\mu\nu} + \Pi^{\mu\nu}. \tag{4.4}$$

Based on the symmetries of our theory, we derived several constraints on $\Pi^{\mu\nu}$ in (3.13), (3.15), and (3.19). In particular $\Pi^{\mu\nu}$ has to be symmetric, traceless, and orthogonal to u^μ. This means in $2+1$ dimensions, $\Pi^{\mu\nu}$ has $9-3-3-1=2$ degrees of freedom (we lose 3 from symmetry, 3 from orthogonality with u^μ, and 1 from being traceless). Write these two degrees of freedom as abstract new variables Π and B, replacing our old "higher-order" variables of v_x and v_y. For our simulation, we choose Π and B to satisfy

$$\begin{aligned}\Pi_{xx} &= \Pi + (2 + u_x^2 + u_y^2)B \\ \Pi_{yy} &= \Pi - (2 + u_x^2 + u_y^2)B.\end{aligned} \tag{4.5}$$

We can rewrite our equations as

$$M\dot{u} = b(u) \tag{4.6}$$

where M is a 5×5 matrix, u is vector of all our variables, and $b(u)$ is a function that involves only spatial derivatives of the variables that can be computed using spectral decomposition on a grid. This provides a point-wise linear system of five variables, which we numerically solve by plugging in values for M and b, and then invert the matrix numerically in MATLAB. In order to make this computation faster, we first solved this problem for general M, examining the matrix for sparseness, and used this general formula at each time step numerically.

One final problem remains: we need to specify five initial conditions for the differential equations to evolve. But physically the system is described by the temperature and velocity. The initial conditions for Π and B must somehow be determined from the velocity and temperature fields. If we use inconsistent Π or B's, then the solution will be unphysical.

We implemented this by solving for Π and B in a lower order hydrodynamic theory given some initial conditions of u_x and u_y, and then used the result as an initial condition for Π and B. Moreover, we checked that Π and B were always physical by checking that they satisfied their definitions from (4.5) at each time step, when we had access to time derivatives of u^μ. This is known as solving the constitutive relations, and serves as a check to the hydrodynamics solution.

4.3.1 Scaling and Conformal Symmetry

In our numerical analysis of hydrodynamics, we exploited the conformal symmetry of the fluid to a great deal. Note the conformal symmetry of our system implies that are no dimensionful quantities in our field theory. Thus, everything is scale-invariant – the physics is determined only by ratios of dimensionful quantities.

For numerical purposes, if we scaled the size of the box we put our fluid in by a factor of c, and then scaled the temperature by a factor of $\frac{1}{c}$, the physics should remain the same, up to the time being scaled a factor of c as well.

4.4 Turbulent Flow

By experimenting with initial conditions, we found suitable ones such that we generated turbulent flow. We saw that our relativistic system quickly relaxed to nonrelativistic speeds, so in order to speed up the calculation, we dropped all terms of v^2 and higher in our expression for M in (4.6). We verified that this approximation is very robust.

Recall in Section 3.3.1 we described the Kolmogorov power scaling law of $k^{-\frac{5}{3}}$ for turbulent fluids. In Figure 4-1, we plot our power spectrum as a function of the Fourier mode k. A comparison is shown with a $k^{-\frac{5}{3}}$ power law. Note that the Kolmogorov scaling has been observed in ideal hydrodynamics by F. Carrsco et al.

in [28], and previous independently by A. Adams and P. Chesler in [30].

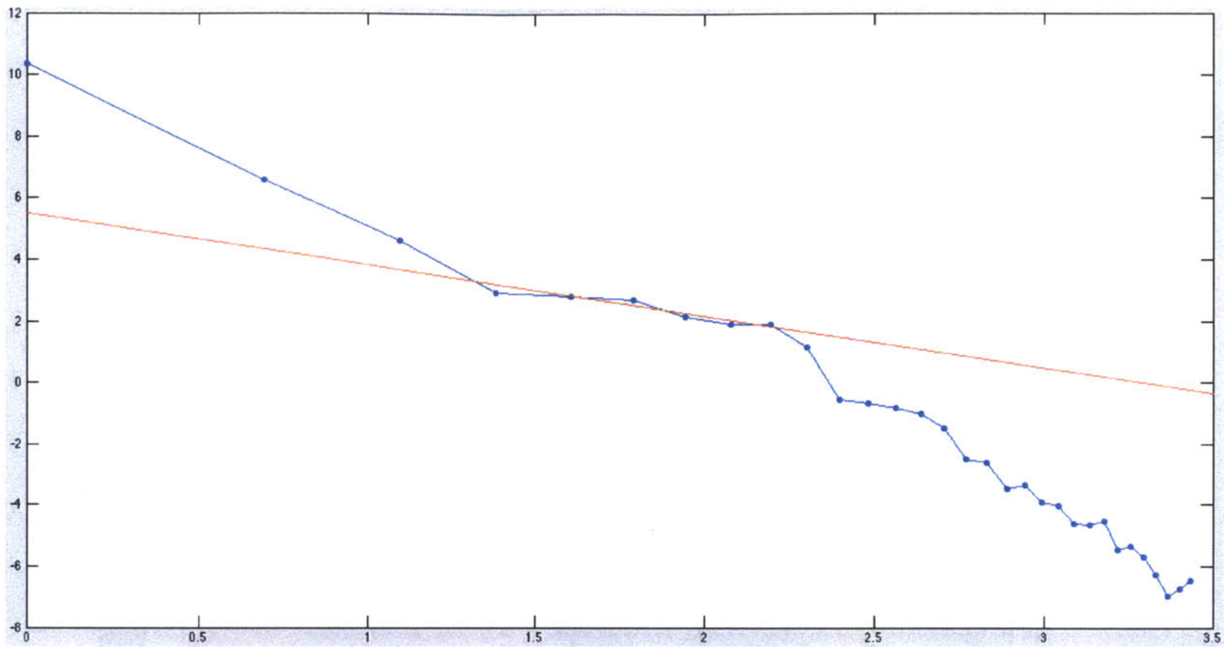

Figure 4-1: A log-log plot of power spectrum vs. frequency for a turbulent flow. The blue plot shows our turbulent fluid and the red plot shows a $k^{-5/3}$ scaling. Initial conditions were a sinusoidal wave in x of amplitude 0.1 with perturbations in both x and y. The simulation was done on a 64×64 grid with $T = 500$ and $L = 40$, screenshot at time $t = 2270$. All terms of order v^2 and higher were dropped to speed up the calculation.

The figure shows the Kolmogorov scaling found in a part of the plot, but has a few odd features. The first is that the low-energy IR modes do not vanish. This is possibly due to the small number of oscillations in the plot – because we only have about two wavelengths in the box, the low frequency modes are very difficult to resolve. The second is that we do not observe a k^{-3} scaling in the UV. In Figure 4-2, we plot the vorticity of the fluid on its grid using color coding, defined as

$$\omega = \partial_y u_x - \partial_x u_y. \tag{4.7}$$

In the plot of the vorticity, our turbulent behavior is clear.

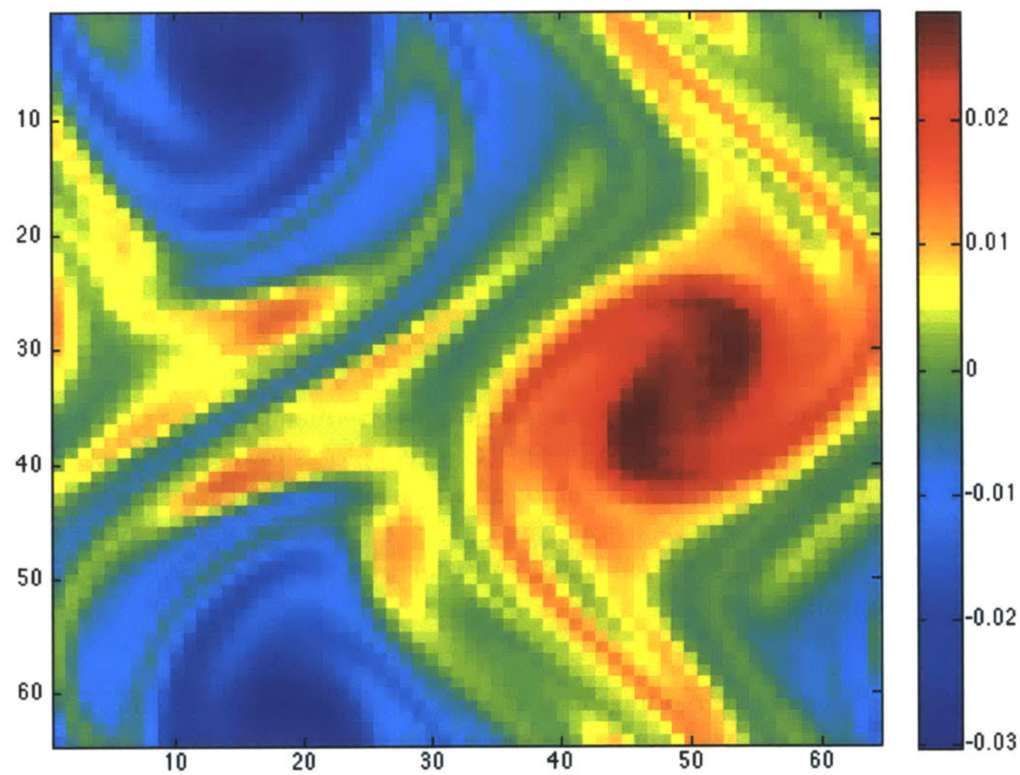

Figure 4-2: A plot of the vorticity of the turbulent fluid. Initial conditions were a sinusoidal wave in x of amplitude 0.1 with perturbations in both x and y. The simulation was done on a 64×64 grid with $T = 500$ and $L = 40$, screenshot at time $t = 2270$. All terms of order v^2 and higher were dropped to speed up the calculation.

Chapter 5

The Fluid/Gravity Correspondence

In this section, we will outline the fluid/gravity correspondence. We follow the presentation in [2] [3] [4].

5.1 Overview

Consider the metric for a black brane in asymptotic AdS space at temperature T in infalling coordinates, (2.15).

$$ds^2 = 2dvdr - r^2 f(br)dv^2 + r^2 d\vec{x}^2 \tag{5.1}$$

where $f(r) = 1 - \frac{1}{r^3}$ and $b = \frac{3}{4\pi T}$.

Note that this metric has an $SO(2,1)$ symmetry amongst the v and $\vec{x}$ coordinates. Thus if we apply a boost u^μ to the solution, the metric remains a solution to the Einstein Field Equations. Note that this is functionally equivalent to imagining the black brane from the point of view of an observer traveling with some velocity u^μ relative to the brane.

$$ds^2 = -2u_\mu dx^\mu dr - r^2 f(br) u_\mu u_\nu dx^\mu dx^\nu + r^2 P_{\mu\nu} dx^\mu dx^\nu. \tag{5.2}$$

with $P_{\mu\nu}$ defined in (3.21).

By Lorentz invariance (5.2) is still clearly a solution to the Einstein Field Equations, given by

$$R_{IJ} - \frac{1}{2} g_{IJ} R + \Lambda g_{IJ} = 0. \tag{5.3}$$

However, if we allow u^μ to vary along x^μ(promoting it to a function of x^μ), the metric will clearly no longer satisfy the Einstein Field Equations generically. An interesting question to ask is: what constraints must the u^μ's satisfy in order to solve Einstein Field Equations?

We can plug in (5.2) into the Einstein Field Equations and explicitly find the constraints. However, if we assume that u^μ satisfies the hydrodynamic assumption in (3.2), we can expand u^μ order by order in our formal parameter ϵ. We then obtain constraints on u^μ that must be satisfied order by order in ϵ (or more schematically, in ∂).

Remarkably, the perturbative constraints of the u^μ reduce to none other than the equations of hydrodynamics with certain transport coefficients.

This means that given a (perturbative) solution of relativistic hydrodynamics in $2+1$ dimensions, we can construct a (perturbative) solution to Einstein gravity in $3+1$ dimensions via the fluid/gravity correspondence. We will outline the details of this calculation in Section 5.2.

5.2 Perturbative Expansion of the Metric

In this section we repeat a calculation done in [6]. Much of our presentation in this section will follow his. Notably, our results differ from [6] in a few subtle and nontrivial parts. We've reproduced the entire calculation and understand the cause of the minor but important discrepancies.

5.2.1 General Strategy

We study the metric by doing a perturbative expansion in derivatives of the velocity field. But before we begin, we need to completely fix the gauge of our metric (in other words, restricting our coordinate choice to remove redundancies). We will chose infalling coordinates as our gauge, where the n^{th}-order metric is of the form

$$ds^2 = \frac{k_n}{r^2} u_\mu u_\nu dx^\mu dx^\nu - 2h_n u_\mu dx^\mu dr - r^2 h_n P_{\mu\nu} dx^\mu dx^\nu - \frac{2}{r}(j_n)_\nu u_\mu dx^\mu dx^\nu + r^2 (\alpha_n)_{\mu\nu} dx^\mu dx^\nu \tag{5.4}$$

with j^μ and $\alpha^{\mu\nu}$ are orthogonal to u^μ and $\alpha^{\mu\nu}$ is traceless.

The goal is to solve for h, j, k and α at each order perturbatively (with zeroth order given by 5.2).

To find the n^{th}-order solution, we took the $(n-1)^{\text{th}}$-order metric, and added our new h, j, k, and α terms. We plugged this solution into Einstein Field Equations, and evaluated it *at a local rest frame.* In other words, we set the fluid-velocity u^μ to be $(1, 0, 0)$. Effectively, we are considering the equations at a single convenient point in the manifold (which must always exist by simply performing a local Lorentz transformation).

The Einstein Field Equations separated into two types. Some of the equations reduced to equations of $(n-1)^{\text{th}}$ order hydrodynamics. Note that they reduced to $(n-1)$ not n because the equations of motion by definition have n derivatives. Therefore our expression for $\partial_\mu T^{\mu\nu} = 0$ has n derivatives, so the stress-energy tensor must have only one lower order derivatives. The others gave the constraints for the next order in perturbation theory.

The equations that come out are, of course, not obviously Lorentz invariant, given that we evaluated it at a particular point. However, we can "bring back" the Lorentz invariance by reconstructing the tensors made up of covariant terms that, when reduced to the local frame, give the results we see.

5.2.2 Zeroth Order

The zeroth order solution for the metric is straightforward. Because we do not introduce any corrections yet, the metric is simply

$$ds^2 = -2u_\mu dx^\mu dr - r^2 f(br) u_\mu u_\nu dx^\mu dx^\nu + r^2 P_{\mu\nu} dx^\mu dx^\nu. \tag{5.5}$$

Schematically this makes sense – in the zeroth order metric solution, we ignore all terms with any derivatives on u^μ, so we can roughly imagine this meaning the solution thinks it has a constant u^μ.

5.2.3 First Order

If we perform the procedure in Section 5.2.1, we find that the constraints on our h, j, k, and α at n^{th}-order reduce to equations

$$\begin{aligned} \frac{1}{r^4}\frac{d}{dr}(r^4 h_n'(r)) &= S_h^{(n)}(r) \\ \frac{d}{dr}\left(-\frac{2}{r}k_n(r) + (1-4r^3)h_n(r)\right) &= S_k^{(n)}(r) \\ \frac{r}{2}\frac{d}{dr}\left(\frac{1}{r^2}\frac{d}{dr}\vec{j}_n(r)\right) &= \vec{S}_j^{(n)}(r) \\ \frac{d}{dr}\left(-\frac{1}{2}r^4 f(r)\frac{d}{dr}\alpha_n^{ij}(r)\right) &= \mathbf{S}_\alpha^{(n)}(r) \end{aligned} \tag{5.6}$$

When solving these equations, we need to impose boundary conditions that match the sourceless equations, typically achieved by setting the homogenous terms to 0. We will explore the boundary conditions in more detail in Appendix A.2.

At first order, we find the sources in (5.6) in the local rest frame to be

$$\begin{aligned} S_h^{(1)} &= 0 \\ S_k^{(1)} &= -4r\partial_i\beta_i \\ \vec{S}_j^{(1)} &= -\frac{1}{r}\partial_\nu\vec{b} \\ \mathbf{S}_\alpha^{(1)} &= 2r\sigma \end{aligned} \tag{5.7}$$

with

$$\sigma_{ij} = \frac{1}{2}(\partial_i\beta_j + \partial_j\beta_i - \frac{1}{2}\delta_{ij}\partial_k\beta_k) \tag{5.8}$$

If we solve these equations we then get that the local solutions for our metric components become

$$\begin{aligned}
h_1(r) &= 0 \\
k_1(r) &= r^3 \partial_i \beta_i \\
\vec{j}_1(r) &= r^2 \partial_\nu \vec{b} \\
\alpha_1(r) &= 2\sigma F(r)
\end{aligned} \tag{5.9}$$

with

$$F(r) = -\frac{\sqrt{3}}{3} \arctan\left(\frac{\sqrt{3}}{3}(2r+1)\right) + \frac{1}{2}\log\left(1 + \frac{1}{r} + \frac{1}{r^2}\right) + \frac{\sqrt{3}\pi}{6}. \tag{5.10}$$

The remaining equations from the Einstein field equations reduce to hydrodynamics equations:

$$\begin{aligned}
\partial_\nu b &= \frac{1}{2}\partial_i \beta_i \\
\partial_i b &= \partial_\nu \beta
\end{aligned} \tag{5.11}$$

5.2.4 Second Order

If we perform the same analysis to second-order in the metric, we get more complicated results. Our source terms in (5.6) become[1]

$$\begin{aligned}
S_h^{(2)} &= -\frac{1}{2r^4}\mathcal{S}_6 + F_1(r)\mathcal{S}_7 \\
S_k^{(2)} &= 2\mathcal{S}_3 + \frac{1}{2}\mathcal{S}_5 - \frac{1+4r^3}{2r^3}\mathcal{S}_6 + F_2(r)\mathcal{S}_7 \\
\vec{S}_j^{(2)} &= \frac{1}{2r^2}\mathcal{V}_3 - \frac{1}{2r^2(1+r+r^2)}\mathcal{V}_4 + \frac{-1+r+r^2}{2r^2(1+r+r^2)}(\mathcal{V}_5 - \mathcal{V}_6) \\
\mathbf{S}_\alpha^{(2)} &= F_3(r)(\mathcal{T}_2 + \mathcal{T}_3) + F_4(r)\mathcal{T}_4 + F_5(r)\mathcal{T}_5\sigma
\end{aligned} \tag{5.12}$$

[1]Note that we believe [6] has two typos: we disagree with his $S_k^{(2)}$ and his definition of $F_2(r)$.

where

$$
\begin{aligned}
F_1(r) &= \frac{2(2r+1)F(r)}{r^2(r^2+r+1)^2} - \frac{(r+1)^2}{r^2(r^2+r+1)^2} \\
F_2(r) &= -\frac{2(1+r)(-1+4r^3)F(r)}{r(1+r+r^2)} + \frac{-1+2r+4r^2+4r^3}{r(1+r+r^2)} \\
F_3(r) &= 2rF(r) - \frac{2r(1+r)}{1+r+r^2} \\
F_4(r) &= -\frac{r}{2}F(r) - \frac{1}{2(1+r+r^2)} \\
F_5(r) &= \frac{3r}{2}F(r) + \frac{1-2r-2r^2}{2(1+r+r^2)}
\end{aligned}
\tag{5.13}
$$

and $F(r)$ is as in (5.10), and finally

$$
\begin{aligned}
\mathcal{S}_3 &= \partial_\nu \partial_i \beta_i \\
\mathcal{S}_4 &= \partial_\nu \beta_i \partial_\nu \beta_i \\
\mathcal{S}_5 &= (\partial_i \beta_i)^2 \\
\mathcal{S}_6 &= (\epsilon_{ij} \partial_i \beta_j)^2 \\
\mathcal{S}_7 &= \sigma_{ij} \sigma_{ij} \\
\mathcal{V}_{3i} &= \partial_i \partial_j \beta_j \\
\mathcal{V}_{4i} &= \partial^2 \beta_j \\
\mathcal{V}_{5i} &= \partial_\nu \beta_i \partial_j \beta_j \\
\mathcal{V}_{6i} &= \partial_\nu \beta_j \partial_i \beta_j \\
\mathcal{V}_{7i} &= \partial_\nu \beta_j \partial_j \beta_i \\
\mathcal{T}_{2ij} &= \partial_\nu \sigma_{ij} \\
\mathcal{T}_{3ij} &= \partial_\nu \beta_i \partial_\nu \beta_j - \frac{1}{2}\delta_{ij}(p_\nu \beta_k)^2 \\
\mathcal{T}_{4ij} &= \partial_k \beta_i \partial_k \beta_j - \frac{1}{2}\delta_{ij}(\partial_k \beta_l)^2 \\
\mathcal{T}_{5ij} &= \partial_i \beta_k \partial_j \beta_k - \frac{1}{2}\delta_{ij}(\partial_k \beta_l)^2.
\end{aligned}
\tag{5.14}
$$

The remaining equations from the Einstein field equations reduce to first-order hydrodynamic equations and constraints:

$$\begin{aligned}
\partial_\nu b &= \frac{1}{2}\partial_i\beta_i - \frac{1}{3}\mathcal{S}_7 \\
\partial_i b &= \partial_\nu\beta - \frac{1}{3}\mathcal{V}_4 - \frac{2}{3}\mathcal{V}_5 + \frac{2}{3}\mathcal{V}_6.
\end{aligned} \tag{5.15}$$

The only remaining thing to do now is to make our expressions covariant. This can be done by considering what expressions will lead to (5.14) in a local rest frame. The result is given below.[2]

$$\begin{aligned}
\mathcal{S}_3 &= u^\mu\partial_\mu\partial_\nu u^\nu - \mathcal{D}u^\mu\mathcal{D}u_\mu \\
\mathcal{S}_4 &= \mathcal{D}u_\mu\mathcal{D}u^\mu \\
\mathcal{S}_5 &= (\partial_\mu u^\mu)^2 \\
\mathcal{S}_6 &= (\epsilon^{\mu\nu\lambda}u_\mu\partial_\nu u_\lambda)^2 \\
\mathcal{S}_7 &= \sigma^{\mu\nu}\sigma_{\mu\nu} \\
\mathcal{V}_{3\mu} &= P_\mu^\sigma(\partial_\sigma\partial_\nu u^\nu - \mathcal{D}u_\nu\partial_\sigma u^\nu) \\
\mathcal{V}_{4\mu} &= P_\mu^\sigma P^{\rho\nu}\partial_\rho\partial_\nu u_\sigma \\
\mathcal{V}_{5\mu} &= P_\mu^\sigma\partial_\nu u^\nu\mathcal{D}u_\sigma \\
\mathcal{V}_{6\mu} &= P_\mu^\sigma\mathcal{D}u_\nu\partial_\sigma u^\nu \\
\mathcal{T}_2^{\mu\nu} &= \Pi_{\alpha\beta}^{\mu\nu}(\mathcal{D}\partial^\alpha u^\beta) \\
\mathcal{T}_3^{\mu\nu} &= \Pi_{\alpha\beta}^{\mu\nu}(\mathcal{D}u^\alpha\mathcal{D}u^\beta) \\
\mathcal{T}_4^{\mu\nu} &= \Pi_{\alpha\beta}^{\mu\nu}(\partial_\rho u^\alpha\partial^\rho u^\beta + \mathcal{D}u^\alpha\mathcal{D}u^\beta) \\
\mathcal{T}_5^{\mu\nu} &= \Pi_{\alpha\beta}^{\mu\nu}(\partial^\alpha u^\rho\partial^\beta u_\rho).
\end{aligned} \tag{5.16}$$

Finally our solutions for h_2, k_2, j_2, and α_2 must be done numerically with appropriate boundary conditions, since the equations are far too difficult to solve analytically.

[2]We believe [6] has a minus sign error in his expression for $\mathcal{S}_6$ here, as well as a trivial index typo in $\mathcal{S}_3$.

At second order, the final metric we obtain is

$$\begin{aligned} ds^2 = & - 2u_\mu dx^\mu dr - r^2 f(br) u_\mu u_\nu dx^\mu dx^\nu + r^2 P_{\mu\nu} dx^\mu dx^\nu \\ & r\partial_\lambda u^\lambda u_\mu u_\nu dx^\mu dx^\nu - r u^\lambda \partial_\lambda (u_\mu u_\nu) dx^\mu dx^\nu + 2r^2 bF(br)\sigma_{\mu\nu} dx^\mu dx^n \\ & \frac{k_2(br)}{b^2 r^2} u_\mu u_\nu dx^\mu dx^\nu - 2b^2 h_2(br) u_\mu dx^\mu dr - r^2 b^2 h_2(br) P_{\mu\nu} dx^\mu dx^\nu \\ & - \frac{2}{br} u_\mu dx^\mu dx^\nu + b^2 r^2 (\alpha_2)_{\mu\nu}(br) dx^\mu dx^\nu \end{aligned} \tag{5.17}$$

Equation (5.15) in its covariant forms reduces to hydrodynamics with a stress energy tensor of

$$\begin{aligned} T^{\mu\nu} = \quad & \frac{1}{2}\left(\frac{4\pi T}{3}\right)^3 (\eta^{\mu\nu} + 3u^\mu u^\nu) - \left(\frac{4\pi T}{3}\right)^2 \sigma^{\mu\nu} \\ & + \frac{1}{18}\left(\frac{4\pi T}{3}\right)((\sqrt{3}\pi - 9\log(3) + 18)\Sigma_1^{\mu\nu} \\ & + (2\sqrt{3}\pi - 9\log(3))\Sigma_2^{\mu\nu}) \end{aligned} \tag{5.18}$$

with $\sigma^{\mu\nu}$, $\Sigma_1^{\mu\nu}$, and $\Sigma_2^{\mu\nu}$ defined the same as in (3.28). For convenience we will redefine T' to be $\frac{4\pi T}{3}$ (in our numerical simulations we dealt with the variable T' instead of T).

Chapter 6

Numerical Methods in the Fluid/Gravity Correspondence

The fluid/gravity correspondence makes a bold claim: to a solution of second order hydrodynamics, plug into (5.17) and the result will satisfy the Einstein Field Equations to second-order in derivatives of the velocity.

We explicitly did this by finding a solution to hydrodynamics, and plugging it in to (5.17). We did this to zeroth-order, first-order, and second-order.

6.1 Error Calculation

In order to determine how well the metric satisfied the Einstein Field Equations, we need some probe for the error. This is a subtle point, because the error is an intrinsically local quantity defined at each point in our spacetime manifold. If we define the Einstein tensor

$$G_{IJ} = R_{IJ} - \frac{1}{2}Rg_{IJ} + \Lambda g_{IJ} \tag{6.1}$$

we expect G_{IJ} to roughly vanish at every point. The fluid/gravity correspondence says that at second-order in the metric, we expect

$$G_{IJ} \sim (\partial u)^3. \tag{6.2}$$

The proper measure of the error of the metric would involve the difference between our metric and the actual metric

$$\Delta \sim g_{IJ}^{\text{actual}} - g_{IJ}^{(n)} \tag{6.3}$$

where g_{IJ}^{actual} is exact metric at all orders, and $g_{IJ}^{(n)}$ is the metric at n^{th} order in our expansion. Unfortunately this quantity is not feasible for us to calculate given that we do not know g_{IJ}^{actual}.[1] So instead we can calculate other quantities that serve as heuristic measures for the error of the metric.

A reasonable quantity to compute is the trace of the Einstein tensor $G_{IJ}g^{IJ}$ and see the average value it takes over manifold at a particular timeslice. Note that this is equivalent to computing the value of

$$\Delta = R + 12 \tag{6.4}$$

across the manifold, which should vanish for an exact solution.

Other reasonable quantities include integrating the l^2 norm of the Einstein tensor across the manifold along a timeslice. This is a slightly subtle point, however, because the quantity is UV divergent – at the boundary of the space, we pick up a divergence that comes the metric containing a $\frac{1}{r^2}$ term which blows up at $r = 0$. We can of course regulate this divergence through, for instance, a hard UV cutoff, but this is a difficult regulator to consistently implement across different solutions. Moreover, numerical errors that come when dealing with very large numbers may give us some trouble. So instead, we dealt with the finite trace of the Einstein tensor.

[1] If we solve the gravity numerically, we can in fact do this computation. This is done in [29].

One more quantity that deserves consideration is

$$\Delta \sim \frac{|E_{IJ}|}{|g_{IJ}|} \tag{6.5}$$

where we take the determinant of each tensor. We have not investigated the behavior of this error, but perhaps will in future work.

6.2 Results

Figure 6-1 shows a plot of the trace of the maximum value of the Einstein tensor over r, averaged over x and y for a simulation at $T' = 1$. Figure 6-2 shows the same plot with identical initial conditions but for a hydrodynamic simulation running at $T' = 10$. The initial conditions were 1000 modes with frequency 3 with random amplitudes normalized to 0.1, and random phase shifts in a box of physical size 100. The grid was a 32×32 grid in x and y with 32 collocation points in the r direction in a Chebyshev grid.

For both figures, the first order metric is a substantially better solution than the zeroth order metric, but the second order is questionable. For $T' = 1$, we obtain roughly similar levels of error for first and second order; for $T' = 10$ however, the second order metric contains worse error scaling. We are not sure as to the reason for our second order metric not giving us the desired results, and are in the process of investigating this.

Note that in both figures, at all orders, the error decays as a function of time. This is the desired behavior: we inputed a solution of second order hydrodynamics to the system, which includes dissipative terms. The dissipative terms will eventually relax the hydrodynamics simulation into equilibrium, at which point the fluid will be stationary and the metric will reduce to (2.15) which of course is an exact solution with zero error.

We are still investigating the reasons of the second order failure to help lower the error. Because the calculation is very involved, it is possible that the discrepancy is

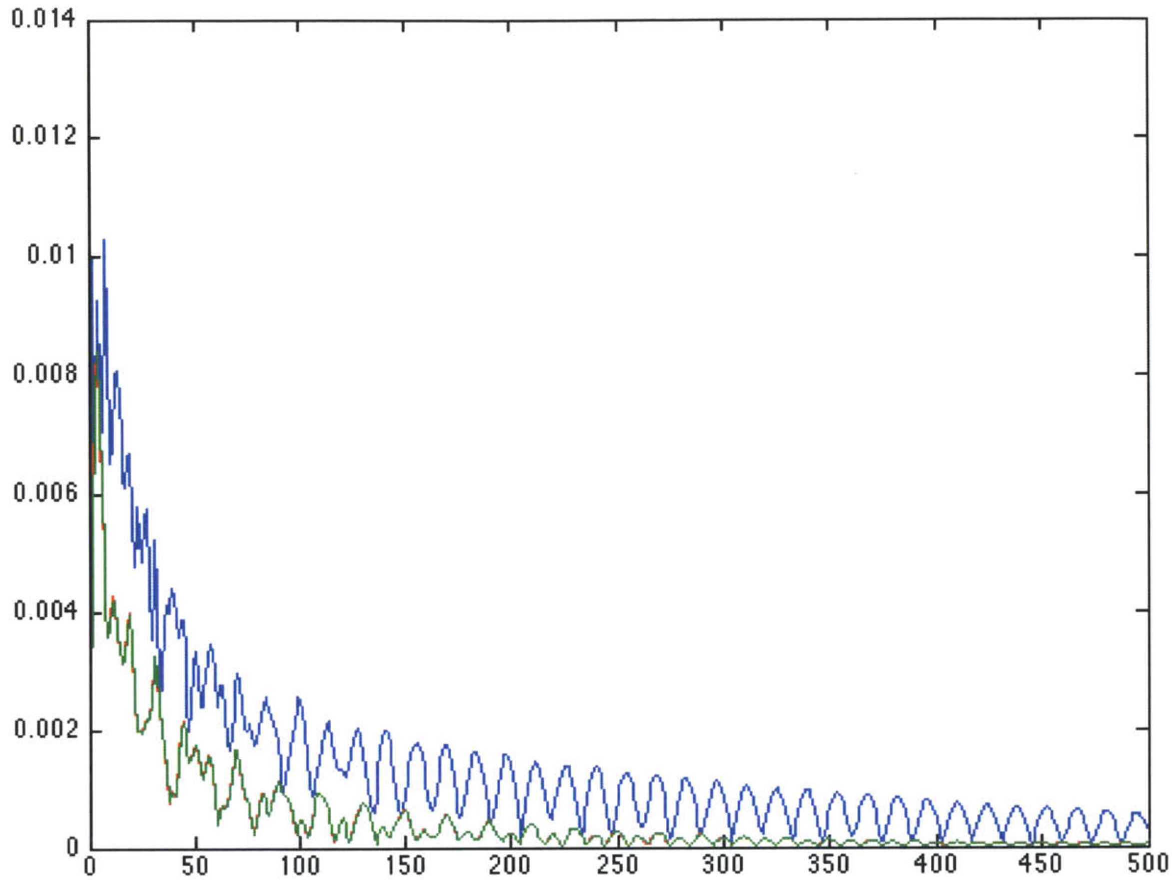

Figure 6-1: Plot of error as a function of time. The blue corresponds to zeroth order; the red corresponds to first order; and the green corresponds to second order. Initial conditions were 1000 random waves with frequency 3 and normalized amplitude 0.1 in a box at $T' = 1$.

due to an error in the MATLAB code.

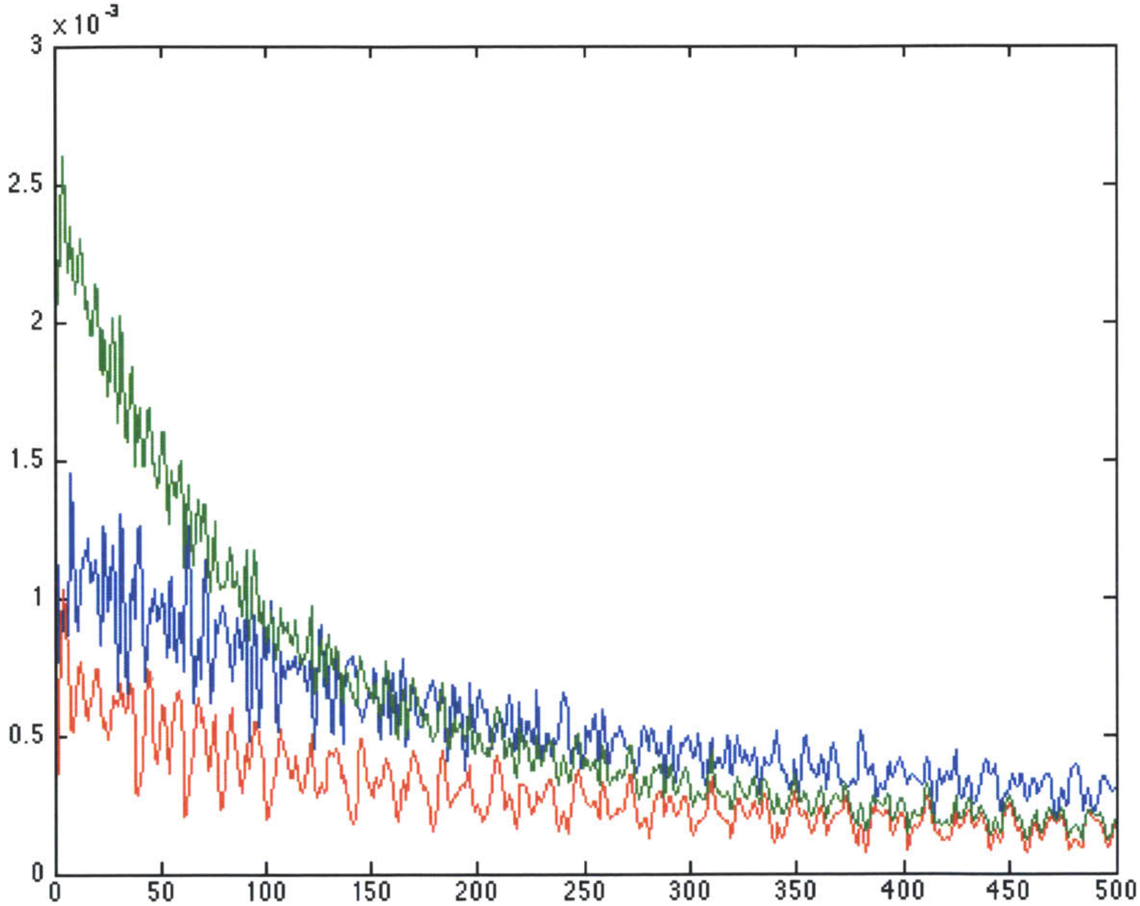

Figure 6-2: Plot of error as a function of time. The blue corresponds to zeroth order; the red corresponds to first order; and the green corresponds to second order. Initial conditions were 1000 random waves with frequency 3 and normalized amplitude 0.1 in a box at $T' = 10$.

Chapter 7

Conclusions

Holographic duality gives us a very powerful tool to study strongly interacting systems by looking at their gravitational dual. Heuristically speaking, when the non-gravitational system is "difficult", the gravitational dual will usually reduce to Einstein gravity in asymptotic AdS space. We can take the low-energy limit of a non-gravitational field theory to be hydrodynamics, and consider the dual of the hydrodynamic theory. This is known as the fluid/gravity correspondence.

We took a solution of the fluid/gravity correspondence in [6] and numerically implemented the calculation. First we solved relativistic hydrodynamics in $2+1$ dimensions and analyzed turbulent systems, a system that has not been extensively studied in the literature to our knowledge. We then took our solution of hydrodynamics and plugged it into the metric solution at zeroth, first, and second order in derivatives of the hydrodynamics expansion. We then calculated how close these metrics were to the "true" metric by analyzing the trace of the Einstein tensor and how it varied over our manifold.

Our results are two-fold: First we have shown that the zeroth and first order metric solutions are clear solutions of the Einstein Field Equations to the accuracy that they advertise. We have implemented this numerically, which is a very nontrivial calculation. We also attempted this calculation at second order, but the results do not yet behave as expected. However, given the proof of concept at zeroth and first order, we are confident that the second order is salvageable and there is some error in our calculation.

Finally we corrected several mistakes currently in the literature [6] about the fluid/gravity correspondence is $2+1$ dimensions.

See Fig. 7-1 for a summary.

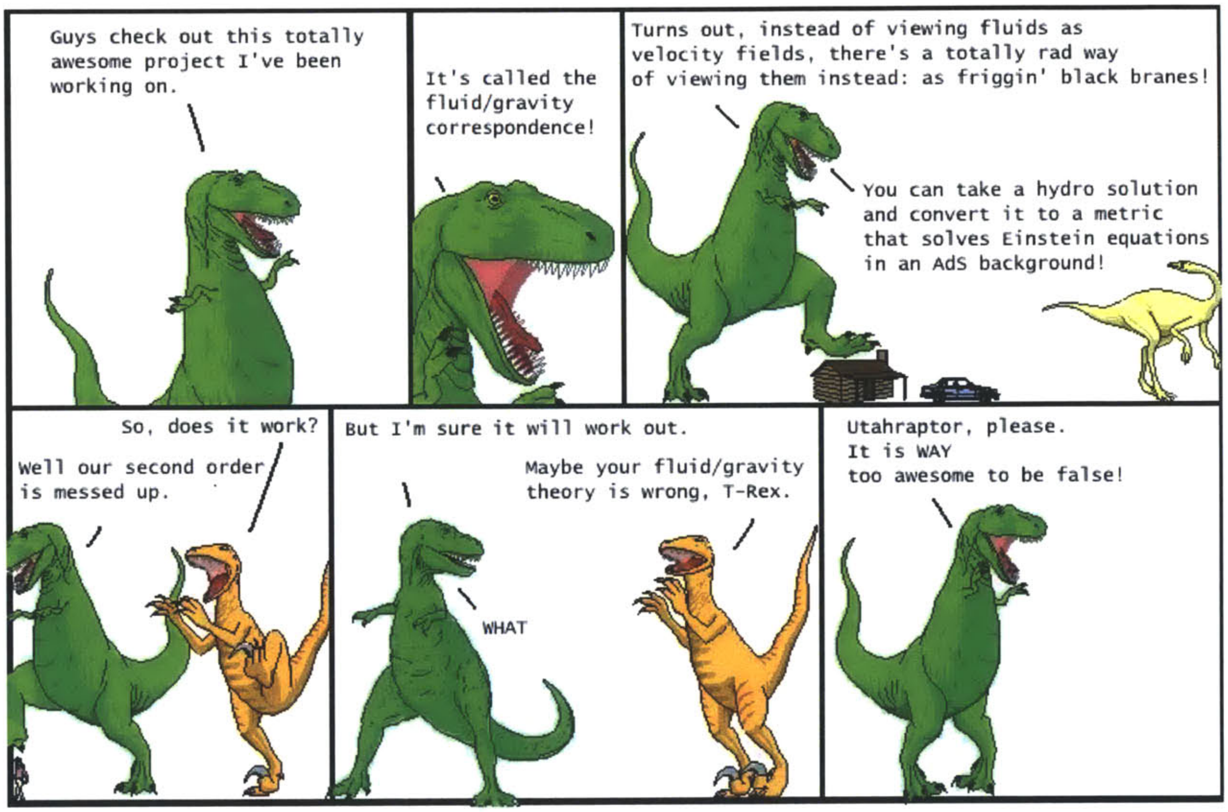

Figure 7-1: A summary of our results.

Appendix A

Subtleties

A.1 Raising and Lowering Indices

When dealing with the bulk metric (e.g. (5.17)), we've raised and lowered indices with Greek indices without much detail in the meaning. This is a subtle point, and in this appendix we will flesh out what we mean.

In the bulk metric, note that there are two relevant symmetries going on: one is the $3+1$ diffeomorphism invariance, and the other is the $2+1$ $SO(2,1)$ symmetry defined on the boundary of the spacetime. As noted in the Section 1.1, Greek indices refer to indices defined on the boundary (t, x, and y), and capital Latin indices refer to all four indices in the bulk.

Thus, when dealing with expressions such as those in (5.17), when raising or lower Greek indices, we use the *boundary* metric $\eta_{\mu\nu}$, not the full metric, g_{IJ}. This is a nontrivial point because we emphasize that we are *not* raising or lowering with the full metric and then restricting ourselves to the t, x, and y coordinates as this a very unnatural operation to consider.

We also do not raise or lower lowercase Latin indices, referring to x and y. The relevant metric there is δ_{ij}.

A.2 Imposing Regularity

In (5.6), we solve the partial differential equations for the sources, and need to impose some boundary conditions on the metric components h, k, j, and α. We can write the solutions to (5.6) as a sum of an inhomogenous solution and a homogenous, source-free solution.

In an arbitrary solution, the metric coefficient for α is singular at the horizon, because the homogenous solution has a divergent logarithmic piece. Our boundary conditions will then be the unique solution for α that gives a metric that is regular everywhere (including the horizon).[1]

For the h, k, and j equations, however, it is enough to say that the homogenous modes must vanish.

[1]Note that this is again a disagreement with [6].

Appendix B

Numerical Methods for Differentiation

In this section I will discuss numerical methods used for differentiating on a grid and solving differential equations.

B.1 Finite Differencing

The most obvious way to define derivatives on a grid is through finite differencing. Given a function $f(x)$ defined on a lattice grid with width L (for simplicity we make the grid one dimensional, but generalizations to d dimensions should be straightforward), we can define the derivative to be

$$f'(x_0) = \frac{f(x_0 + L) - f(x_0)}{L}. \tag{B.1}$$

This is known as a finite differencing derivative to first order. But we can do a bit better by averaging

$$\begin{aligned} f'(x_0) &= \frac{1}{2}\left(\frac{f(x_0 + L) - f(x_0)}{L} + \frac{f(x_0) - f(x_0 - L)}{L}\right) \\ &= \frac{1}{2}\left(\frac{f(x_0 + L) - f(x_0 - L)}{L}\right). \end{aligned} \tag{B.2}$$

This is finite differencing to second order. We can extend this to define higher order derivatives, for instance

$$f'(x_0) = \frac{f(x_0 - 2L) - 8f(x_0 - L) + 8f(x_0 + L) - f(x_0 + 2L)}{12L}. \tag{B.3}$$

The intuitive reason for why these derivatives are more and more accurate is that we use more information – instead of relying on the function evaluated at just two grid points, we use several points, using more information and allowing us a better guess for the derivative. We will use this basic strategy in Section B.2 in motivating spectral interpolation and differentiation.

B.2 Spectral Interpolation

The discussion in this section will follow mostly [31] [32] and conversations with Allan Adams and Paul Chesler.[1]

Imagine we write a periodic function as a sum of Fourier modes.

$$f(x) = \sum_{k=-\infty}^{\infty} f_k e^{\frac{2\pi i}{L}kx}. \tag{B.4}$$

with the f_k coefficients suitably defined.

If we then want to take a derivative of this function, we can write it as

$$f'(x) = \sum_{k=-\infty}^{\infty} \left(\frac{2\pi i}{L} k f_k\right) e^{\frac{2\pi i}{L}kx}. \tag{B.5}$$

Thus we can take a derivative by transforming into Fourier space, multiplying by an appropriate factor, and transforming back into position space.

This is the main idea in spectral methods for numerical analysis. To implement

[1] We thank Paul Chesler for allowing us to use his spectral differentiation macros.

spectral methods on a finite grid, we use a discrete Fourier transform instead.

$$f(x) = \sum_{k=0}^{N-1} f_k e^{\frac{2\pi i}{N} kx} \tag{B.6}$$

with

$$f_k = \frac{1}{N} \sum_{i=0}^{N-1} f(i) e^{-\frac{2\pi i}{N} kx}. \tag{B.7}$$

We can generalize this beyond differentiation, and with a non-Fourier basis. The main idea in spectral analysis is that we pick some basis of N reasonable functions that span all functions on a grid of size N, and we write our function as a sum of our basis functions. At this point, we *assume* that the "true" function is the sum of our basis functions defined at all points, including those not on our grid.

Thus, in order to take a derivative, we take the interpolated function (the sum of the N basis functions), and differentiate it honestly (for instance, analytically). Then we evaluate the derivative at any point (including not at collocation points).

We can use spectral interpolation beyond simply differentiation. Instead of performing our operation on a grid, we assume the continuous form of the function is given by sum of the basis functions that match our function at our collocation points, and compute quantities this way.

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